

Natural Maximal $\nu_\mu - \nu_\tau$ Mixing

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Abstract

The naturalness of maximal mixing between myon- and tau-neutrinos is investigated. A spontaneously broken nonabelian generation symmetry can explain a small parameter which governs the deviation from maximal mixing. In many cases all three neutrino masses are almost degenerate. Maximal $\nu_\mu - \nu_\tau$ -mixing would indicate that the leading contribution to the light neutrino masses arises from the expectation value of a heavy weak triplet rather than from the seesaw mechanism. In this scenario the deviation from maximal mixing is predicted to be less than about 1%.

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Recent measurements of the ν_e and ν_μ yields of atmospheric neutrinos [1] give a strong hint for $\nu_\mu - \nu_\tau$ oscillations characterized by a difference in squared mass $\Delta m_a^2 \approx 5 \cdot 10^{-3} \text{eV}^2$ and large mixing angle $\sin^2 2\vartheta \gtrsim 0.8$. In comparison with the charged fermions the large generation mixing may surprise at first sight. In fact, it can arise quite naturally if the pattern of neutrino masses and mixings is connected to some spontaneously broken generation symmetry. A first investigation of abelian generation symmetries which could explain the hierarchies in the charged fermion masses has revealed [2] that often the generation charge of the myon- and tau-neutrino is the same, $Q(\nu_\mu) = Q(\nu_\tau)$. In consequence the mixing between ν_μ and ν_τ is not suppressed by any small parameter in contrast to the mixing in the charged fermion sector. For such models large $\nu_\mu - \nu_\tau$ -mixing has been predicted [2],[3]. In this approach, all small mass ratios and mixing angles are explained in terms of various powers of a small parameter connected to generation symmetry breaking. All realistic models with generation group $U(1)$ and unification group $SU(5)$ were found to have $Q(\nu_\mu) = Q(\nu_\tau)$. Furthermore, this type of models has recently been investigated in the context of cosmological leptogenesis [4]. Some of the models lead to a cosmological baryon asymmetry in the observed order of magnitude.

Typically, in models with an abelian generation group $\sin^2 2\vartheta$ comes out naturally of order one, but not necessarily very close to one. In this letter we ask: Are there natural models where $\sin^2 2\vartheta = 1 - \epsilon$ with $\epsilon \ll 1$? Such a “maximal $\nu_\mu - \nu_\tau$ -mixing” is compatible with experiment, but a really small parameter, say $\epsilon < 0.1$, is not established. We want to show that maximal $\nu_\mu - \nu_\tau$ -mixing, if found in nature, would imply very interesting consequences for our understanding of possible generation symmetries and their breaking pattern. In particular, generation symmetries explaining naturally $\epsilon \ll 1$ must be nonabelian.

For the left-handed neutrinos of the standard model the smallness of their masses as compared to the charged fermion masses is naturally explained [5] as a consequence of the gauge hierarchy, i.e. the small ratio between the Fermi scale $\sim M_W$ and the unification scale $\sim M$. Since the only allowed mass term is a symmetric neutrino bilinear, it must transform as a triplet under the weak gauge group $SU(2)_L$. No renormalizable operator of this type exists in the standard model (and many extensions of it). Allowed nonrenormalizable operators $\sim \frac{1}{M} \nu \nu d d$ involve the vacuum expectation value d of the Higgs doublet twice. They are suppressed, however, by the inverse of some large scale M , explaining naturally why $m_\nu \sim M_W^2/M$ is much smaller than a charged fermion mass which is $\sim d \sim M_W$ [5, 6, 7, 8]. In many models M is the scale of spontaneous $B-L$ symmetry breaking. The seesaw-mechanism [9] can be understood naturally in this context if the dominant contribution to the nonrenormalizable operator comes from the exchange

of superheavy neutrinos² ν^c . In this case M is associated with the heavy neutrino masses. The seesaw mechanism is, however, only a partial ingredient for an explanation of the smallness of the neutrino masses. We will actually see below that it does not give the dominant contribution to the neutrino masses in case of maximal $\nu_\mu - \nu_\tau$ -mixing. The dominant mass term arises instead from the “induced triplet mechanism”, i.e. the expectation value of a heavy scalar $SU(2)_L$ -triplet [5] which is naturally suppressed³ $\sim M_W^2/M$.

We restrict the discussion here to models where the smallness of the neutrino masses is directly related to the gauge hierarchy. In particular, we assume that all fermions which are not protected by the chiral $SU(2)_L \times U(1)_Y$ gauge symmetry are superheavy and we do not consider generation symmetries which are only broken at the Fermi scale. This implies that there are only three light neutrinos ν_e, ν_μ, ν_τ without any “sterile neutrinos”.

The orders of magnitude $m_f \sim M_W$, $m_\nu \sim M_W^2/M$ for the charged and neutral fermions, respectively, reflect only the very rough structure which is induced by the small scale of $SU(2)_L$ -symmetry breaking. We next pursue the idea that all small quantities in the mass matrices are dictated by symmetry in order to understand the generation fine structure responsible for the substantial splitting within the charged or neutral fermion masses. A generation symmetry can differentiate between the electron and the τ or the up and the top quark since those may transform differently. The spontaneous breaking of the generation group induces a small parameter λ (or several such parameters) [12], [13], [14], [15]. It corresponds to the ratio M_G/M with M_G a characteristic scale for the spontaneous breaking of the generation group. We deal here with a fine structure around the unification scale with typical orders of magnitude $\lambda = M_G/M \approx 10^{-2} - 10^{-1}$. Depending on the generation charges of the various fermion bilinears which appear in the fermion mass matrices a certain number p of generation symmetry breaking operators is needed in order to construct a singlet. In consequence, the corresponding matrix element of a charged fermion mass matrix is proportional $\sim \lambda^p M_W$. The phenomenologically required powers of λ have been discussed systematically in [16]. We may call the doublet which is responsible for the spontaneous symmetry breaking of $SU(2)_L$ in the limit of unbroken generation symmetry the “leading doublet”. Typically, it only contributes to the top quark mass [14],[15]. Due to generation symmetry breaking, the low mass doublet in the sense of an effective field theory acquires an admixture of other doublets $\sim \lambda^p$.

These ideas were used for an understanding of the mass matrix M_ν for the light neutrinos [2]. For an abelian generation symmetry consistent with

²We use here a notation where all particles are named as left-handed particles. For example, the left-handed antineutrino ν^c is conjugate to the right-handed neutrino.

³See ref. [10] for an application of the induced triplet mechanism [5] for a detailed discussion of triplet potentials in left-right symmetric models. A general discussion of mixing matrices in presence of triplet expectation values can be found in [11].

the structure of the charged fermion mass matrices interesting patterns for M_ν were found and two general lessons became visible: (i) Typically, the mass and mixing pattern for the neutrinos is not similar to the one for the charged fermions. The reason is that generically the $SU(2)_L$ -triplet operators entering M_ν transform differently under the generation group as compared to the $SU(2)_L$ -doublet operators responsible for the charged fermion masses. (ii) The proportionality of the neutrino masses to the squares of the charged fermion masses of the same generation, i.e. $m_{\nu_i}/m_{\nu_j} = m_{f_i}^2/m_{f_j}^2$ is usually not realized.

The discussion in ref. [2] was restricted to an abelian generation symmetry. In this context a large $\nu_\mu - \nu_\tau$ -mixing is often natural, but maximal mixing in the sense of $\epsilon \ll 1$ would be difficult to understand. As an alternative we will see in this note that a non-abelian generation symmetry can indeed explain naturally a neutrino mass pattern where

- (i) the $\nu_\mu - \nu_\tau$ -mixing is maximal, and
- (ii) the mass-squared difference between ν_e and $\nu_A = \frac{1}{\sqrt{2}}(\nu_\mu + \nu_\tau)$ is much smaller than the mass-squared difference with the third neutrino $\nu_B = -\frac{i}{\sqrt{2}}(\nu_\mu - \nu_\tau)$.

Here the pattern of mass eigenstates may either be hierarchical $|m_{\nu_e}| \ll |m_{\nu_A}| \ll |m_{\nu_B}|$ or degenerate. In the degenerate case all three masses are either almost equal and larger than $(\Delta m_a^2)^{1/2}$ or they are typically all of the order $(\Delta m_a^2)^{1/2}$. The simplest generation symmetry leading to such a pattern can be generated from the discrete reflections $R : \nu_e \leftrightarrow \nu_A$ and $T : (\nu_\mu \rightarrow -\nu_\mu, \nu_\tau \rightarrow -\nu_\tau)$.

It is instructive to look first at the muon- and tau-neutrinos neglecting all other particles. The most general (Majorana) mass matrix is symmetric, and for a first approach we also take it to be real

$$M_{\nu,2} = \begin{pmatrix} a - c, & b \\ b, & a + c \end{pmatrix} \quad (1)$$

Almost maximal mixing requires a small ratio $|c/b| \ll 1$ with

$$\sin^2(2\vartheta) = 1 - \frac{c^2}{b^2} + 0 \left(\frac{c^4}{b^4} \right) \quad (2)$$

It is not difficult to conceive a mechanism leading naturally to $|c/b| \ll 1$. An example is a generation symmetry that forbids the diagonal elements in $M_{\nu,2}$. They can then be induced only by spontaneous breaking of such a symmetry, leading to a suppression factor $c/b \sim (M_G/M)^p$. Any symmetry under which ν_μ and ν_τ transform differently while the complex conjugate of the leading $SU(2)_L$ -triplet operator shares the same generation quantum numbers as the bilinear $\nu_\mu \nu_\tau$ will be sufficient for this purpose.

Further restrictions arise if we take into account information about the electron neutrino from solar neutrino observations. An MSW explanation [17] of the solar neutrino puzzle by neutrino oscillations requires that the relevant mass-squared difference $\Delta m_s^2 \approx 5 \cdot 10^{-6} eV^2$ is small as compared to $\Delta m_a^2 \approx 5 \cdot 10^{-3} eV^2$. For three neutrinos this leaves only two possibilities for the pattern of neutrino mass eigenvalues, namely

$$(A) \quad |m_1| \ll |m_2| \approx \sqrt{\Delta m_s^2} \ll |m_3| \approx \sqrt{\Delta m_a^2}$$

or

$$(B) \quad |m_1| \approx |m_2| \approx |m_3| \approx m \quad , \quad \Delta m_s^2 \ll \Delta m_a^2 < m^2$$

We concentrate on the case of small mixing of the electron neutrino with the other neutrinos. The mass matrix $M_{\nu,2}$ (eq.(1)) is then a reasonable approximation for the $\nu_\mu - \nu_\tau$ sector. We denote the eigenvalue with the larger (smaller) absolute size by $m_+(m_-)$ and introduce the ratio

$$R = \frac{m_+^2 - m_-^2}{m_+^2 + m_-^2} = \frac{2|a|\sqrt{b^2 + c^2}}{a^2 + b^2 + c^2} \approx \frac{2|ab|}{a^2 + b^2} \quad (3)$$

The “hierarchical mass pattern” (A) needs R very close to one or

$$a = \pm b(1 + \eta) \quad (4)$$

In this case we need a symmetry explanation for a very small difference $|a| - |b|$, i.e. $|\eta| = 2\sqrt{\Delta m_s^2/\Delta m_a^2}$. Whenever $|\eta|$ exceeds this value, the only alternative is the “degenerate mass pattern” (B). A leading form of the neutrino mass matrix with maximal mixing is

$$M_{max} = \begin{pmatrix} \pm a \pm b & 0 & 0 \\ 0 & a & b \\ 0 & b & a \end{pmatrix} \quad (5)$$

It has two degenerate mass eigenvalues $|m| = |a \pm b|$ and one eigenvalue $|m| = |a \mp b|$. For $(M_{max})_{11} = 0$ ($|a| = |b|$) one has the hierarchical pattern (A). On the other hand, for $|a| \ll |b|$ one finds three almost degenerate values for the neutrino masses⁴ which are substantially larger than $\sqrt{\Delta m_a^2}$

$$\frac{\Delta m_a^2}{m^2} \approx 4 \left| \frac{a}{b} \right| \quad (6)$$

This cosmologically interesting scenario extends qualitatively to the case where a and c are of similar magnitude. Adding the pieces $\sim c$ in the $\nu_\mu - \nu_\tau$ sector induces a contribution $\Delta m_s^2 \approx c^2$. For $c = a$ this yields $m^2/\Delta m_a^2 = \frac{1}{16}\Delta m_a^2/\Delta m_s^2$.

⁴Degenerate neutrino masses without connection to maximal $\nu_\mu - \nu_\tau$ -mixing have been discussed earlier [18], often in a setting where $|a| \gg |b|$.

These considerations can be generalized for complex symmetric matrices. For a $\nu_\mu - \nu_\tau$ -matrix $M_{\nu,2}$ (1) the condition for maximal mixing is now $|Re(ac^*)| \ll [(Re(b^*a))^2 + (Im(b^*c))^2]^{1/2} = 0$. As an example, a leading mass matrix

$$M_A = mY, \quad Y = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & i \\ 0 & i & -1 \end{pmatrix} \quad (7)$$

leads to maximal mixing with a hierarchical mass pattern (A) with eigenvalues $(0, 0, m)$. A matrix of the type (5) with two degenerate eigenvalues and maximal mixing can be written⁵ as

$$M_{max} = bW + a\mathbb{1}, \quad W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad (8)$$

with complex b and a . Here the mass difference obeys $\Delta m_a^2 = 4|Re(ba^*)|$ and for $|Re(ba^*)| \ll |b|^2$ all three masses are approximately degenerate.

Let us next investigate possible generation symmetries which lead to the degenerate mass pattern $M_B = mW$. A global $U(1)$ symmetry with charges (q_e, q_μ, q_τ) for $(\nu_e, \nu_\mu, \nu_\tau)$ where $2q_e = q_\mu + q_\tau$ and $q_\mu \neq q_\tau$ enforces a mass matrix

$$\hat{M}_B = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & 0 & m \\ 0 & m & 0 \end{pmatrix} \quad (9)$$

provided the generation charge of the leading $SU(2)_L$ triplet is $-2q_e$. This generalizes to discrete subgroups as, for example, the Z_3 -symmetry $\nu_e \rightarrow e^{i\varphi}\nu_e$, $\nu_\mu \rightarrow e^{2i\varphi}\nu_\mu$, $\nu_\tau \rightarrow \nu_\tau$, $t \rightarrow e^{-2i\varphi}t$, $\varphi = \frac{2\pi}{3}$. Symmetries enforcing $m_1 = m$ can be found most easily by a unitary change of basis $\nu = U\tilde{\nu}$, i.e.

$$(\nu_e, \nu_\mu, \nu_\tau) = (\nu_e, \nu_A, \nu_B)U^T, \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & -i \\ 0 & 1 & i \end{pmatrix}, \quad (10)$$

which makes the neutrino mass matrix diagonal, $U^T \hat{M}_B U = \text{diag}(m_1, m, m)$. For $m_1 = m$ one has $U^* V^T U^T M_B U V U^\dagger = M_B$ if V is an element of $O(3)$, $V^T V = 1$. Consider now transformations V which form a subgroup of $O(3)$ such that the unit matrix is the only invariant symmetric 3×3 matrix. Then the generation symmetry $\nu \rightarrow S\nu$, $S = UVU^\dagger$, enforces the mass matrix $M_B = mW$ provided the leading $SU(2)_L$ triplet transforms trivially.

A possible (minimal) discrete transformation group can be constructed from the elements

$$R = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 1 & -1 \\ \sqrt{2} & -1 & 1 \end{pmatrix}, \quad T(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\varphi} & 0 \\ 0 & 0 & e^{-i\varphi} \end{pmatrix} \quad (11)$$

⁵Due to the freedom in the choice of phases the matrices M_A and M_{max} are not the most general ones for maximal mixing and given mass eigenvalues.

with $\varphi_n = 2\pi/n$, $n \in \mathbb{N}$, $n \geq 2$. The combination of R and $T_n \equiv T(\varphi_n)$ enforces a matrix of the form M_{max} (eq. (8)) if the triplet transforms trivially. For $n > 2$ it implies furthermore $a = 0$. We note that R corresponds to $\nu_e \leftrightarrow \nu_A = \frac{1}{\sqrt{2}}(\nu_\mu + \nu_\tau)$ with $R^2 = 1$. Both R and $T(\varphi)$ are elements of $O(3)$ obeying the relation $S^T = (UU^T)^* S^{-1} UU^T = WS^{-1}W$ which is equivalent to $V^T V = 1$. The other elements of a discrete generation group can be constructed from multiplications of R and T_n . For example, for $n = 2$ with $T \equiv T_2$, $T^2 = 1$ the list is $(R, T, RT, TR, RTR, TRT, RTRT = TRTR = -W)$. (If one adds an element $I_1 = \text{diag}(i, 0, 0)$ one can enforce the hierarchical pattern where $b = a$.)

The maximal symmetry consistent with a mass matrix M_B can be generated from the $O(3)$ transformations encoded in V (or equivalently, S) and an abelian transformation $\nu \rightarrow e^{i\chi}\nu$, $t \rightarrow e^{-2i\chi}t$. The subgroup of real transformations $S = S^*$ leaves the matrix M_{max} (eq. (8)) invariant. This corresponds to (abelian) rotations in the (ν_e, ν_A) plane. We conclude that maximal mixing with a degenerate mass pattern can arise naturally from a large class of symmetries.

We next have to ask if such a generation symmetry is compatible with a realistic mass pattern for the charged leptons. The leading mass term has the form $l^{cT} d_l^* l$ where d_l denotes the doublet that contributes to the charged lepton mass matrix M_l in leading order. (Typically, d_l is not the leading $SU(2)_L$ -breaking doublet since $m_\tau \ll m_t$.) In general $l^{cT} = (e^c, \mu^c, \tau^c)$ may transform under S differently from $l^T = (e, \mu, \tau)$, i.e. $l \rightarrow Sl$, $l^c \rightarrow S^{(c)} l^c$. The doublet transformation property $d_l^* \rightarrow (S^{(c)T})^{-1} d_l^* S^{-1}$ implies that possible natural mass patterns depend crucially on the representation of l^c . We concentrate here on the case of an $SO(3)$ generation group where l^c and l belong both to triplet representations, i.e. $S^{(c)} = S$. It is then convenient to work in the canonical $SO(3)_G$ basis (10) $\tilde{\nu} = U^\dagger \nu$, $\tilde{l} = U^\dagger l$, $\tilde{l}^c = U^\dagger l^c$ where the leading neutrino mass matrix is diagonal $\tilde{M}_B = U^T M_B U = m \mathbb{1}$ and $SO(3)_G$ is represented by standard real orthogonal matrices V . A leading doublet which only contributes to the τ -mass yields in this basis $\tilde{M}_l = m_\tau Y$ with $Y = U^T \text{diag}(0, 0, 1) U$ given by eq. (7). The matrix Y is traceless and symmetric and the leading doublet should therefore belong to a 5 of $SO(3)_G$. With respect to the $U(1)$ -rotation $T(\varphi)$ (cf. eq. (11))

$$\tilde{T}(\varphi) = U^\dagger T(\varphi) U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & \sin \varphi \\ 0 & -\sin \varphi & \cos \varphi \end{pmatrix} \quad (12)$$

the matrix Y transforms as

$$\tilde{T}(\varphi)^T Y \tilde{T}(\varphi) = e^{-2i\varphi} Y \quad (13)$$

If the doublet transforms as $d_l^* \rightarrow e^{2i\varphi} d_l^*$, only a lepton mass term $\tilde{M}_l \sim Y$ is allowed.

Let us denote the abelian charge related to the rotations $\tilde{T}(\varphi)$ in the (ν_A, ν_B) plane by I_{3G} . The first generation $(\nu_e; e; e^c)$ has $I_{3G} = 0$, the second $(\nu_\mu; \mu; \mu^c)$ carries $I_{3G} = +1$ whereas the third $(\nu_\tau; \tau; \tau^c)$ comes with $I_{3G} = -1$. If the $SU(2)_L$ -doublet d_l^* belongs to the $I_{3G} = 2$ component of an $SO(3)_G$ -5-plet only the τ can acquire a mass! On the other hand, we assume that the leading $SU(2)_L$ -triplet belongs to an $SO(3)_G$ -singlet with $I_{3G} = 0$. The qualitative difference between the hierarchical mass pattern for the charged leptons and the degenerate pattern for the neutrinos can simply be explained by the different transformation properties of the leading doublet and triplet! We emphasize that the relative angle of $\pi/4$ between the basis of eigenstates of I_{3G} and the “canonical $SO(3)_G$ -basis” arises very naturally in this picture. It is the central ingredient for maximal $\nu_\mu - \nu_\tau$ -mixing.

For the maximal mixing scenario the different representations of the doublet and triplet strongly disfavour the generation of the leading contribution to the neutrino mass by the seesaw mechanism [9]. In fact, the Majorana mass matrix for the left-handed neutrinos can be written in the general form [8]

$$M_\nu = M_D^T M_R^{-1} M_D + M_L \quad (14)$$

where M_D is the Dirac mass matrix linking right- and left-handed neutrinos, and M_R is the mass matrix for the heavy or “right-handed” neutrinos. Since M_R transforms as an $SU(2)_L$ -singlet, one expects that its eigenvalues are much larger than the Fermi scale. The seesaw mechanism is realized if M_L can be neglected. On the other hand, M_L arises from the direct coupling of the left-handed neutrinos to a scalar $SU(2)_L$ -triplet⁶. As mentioned above, its expectation value must be proportional to M_W^2/M_t with M_t a large mass scale corresponding to the mass of the scalar $SU(2)_L$ triplet [5]. The degenerate neutrino mass pattern can easily be realized if M_L is the leading contribution to M_ν with the $SU(2)_L$ -triplet transforming as a singlet under the $SO(3)_G$ generation group. In contrast, the Dirac mass term M_D often has a similar structure as the charged lepton mass matrix M_l or the up-type quark mass matrix M_u because it also arises from the coupling to a doublet. As an example we investigate the consequences of a leading-order behavior $M_D \sim M_l \sim \text{diag}(0, 0, 1)$, $M_R \sim M_L \sim W$. Since $W^2 = 1$ one has $M_R^{-1} = \frac{1}{m_R} W$ and $M_D^T M_R^{-1} M_D = 0$. A Dirac mass contribution can therefore only arise at subleading order. For $M_D = \text{diag}(g_e, g_\mu, g_\tau)$, $g_e \ll g_\mu \ll g_\tau$ one finds⁷

$$M_D^T M_R^{-1} M_D = \begin{pmatrix} g_e^2 & 0 & 0 \\ 0 & 0 & g_\mu g_\tau \\ 0 & g_\mu g_\tau & 0 \end{pmatrix} m_R^{-1} \quad (15)$$

⁶In left-right symmetric unification like $SO(10)$ the matrices M_L and M_R are often proportional to each other. This is the case whenever the $SU(2)_L$ -triplet leading to M_L and the $SU(2)_L$ -singlet leading to M_R belong to the same scalar multiplet.

⁷Off-diagonal elements in M_D can also give an important contribution.

This induces a mass split between ν_e on the one side and the two linear combinations of ν_μ and ν_τ on the other side. We conclude that $g_\mu g_\tau / m_R$ should be of the order $\Delta m_s^2 / m$ or smaller. The dominant subleading correction to M_ν lifting the degeneracy between ν_e and ν_A on one side and ν_B on the other side should therefore arise from the expectation value of another $SU(2)_L$ -triplet contributing to M_L or from a correction⁸ to M_R . We have already discussed above that $M_L = bW + a\mathbb{1}$ would account for Δm_a^2 (cf. eq. (6)). This scenario is realized if the next to leading $SU(2)_L$ -triplet expectation value remains invariant under the symmetries R and T or the (ν_e, ν_A) rotations generated by a rotation V in the 1-2 plane such that $S^T S = 1$, $S^* = S$.

We emphasize, however, that the above discussion merely serves as an example. If the unification group does not contain $SU(2)_R$, there is no reason why the left-handed antineutrinos ν^c (which are equivalent to the right-handed neutrinos) should be in the same representation of the generation group as l^c . In consequence, there is then no reason for a proportionality $M_D \sim M_l$. Similarly, if the unification group does not contain $SU(4)_c$, there is no relation between M_D and the mass matrix M_u for the up-type quarks. An investigation of generation symmetries which are compatible with $SU(5)$ reveals [2] that M_D often has a generation structure which is quite different from M_l or M_u . On the other hand, there is also no reason why M_D should be proportional to M_L and M_R . A generation of the maximal mixing matrix M_{max} by the seesaw-mechanism seems therefore unlikely. Furthermore, a proportionality $M_L \sim M_R$ is not expected if ν and ν^c belong to different representations of the generation group.

We next sketch a possible mechanism⁹ for small mass ratios m_μ / m_τ and m_e / m_μ . A mixing of nonleading doublets coupling to muons and electrons with the leading doublet coupling $\sim Y$ is only possible if all abelian subgroups of $SO(3)_G \times U(1)_G$ with charges $\tilde{q} = I_{3G} + \alpha q$ are spontaneously broken. Here q denotes the charge corresponding to the abelian generation group $U(1)_G$ with $q(\nu_e, \nu_\mu, \nu_\tau) = q(e, \mu, \tau) = q_1$, $q(e^c, \mu^c, \tau^c) = q_2$, $q(d) = -(q_1 + q_2)$. The other doublets in the 5 of $SO(3)_G$ have $I_{3G} = (-2, -1, 0, 1)$ as compared to $I_{3G} = 2$ for the doublet d_l^* . This difference in charge carries over to \tilde{q} for arbitrary α . An $SO(3)_G$ breaking ($SU(2)_L$ -singlet) operator with fixed \hat{I}_{3G}

⁸A matrix $M_R = m_R W + s_R \mathbb{1}$, with real m_R, s_R and $|s_R| \ll |m_R|$ generates a Dirac mass contribution $\sim (g_\tau^2 s_R / m_R^2) Y$. Since this contributes to Δm_a^2 and Δm_s^2 in comparable magnitude, the ratio $|s_R / m_R|$ should be small, e.g. $s_R / m_R \approx g_\mu / g_\tau$, or g_τ^2 / m_R must be much smaller than $|b|$. For $g_\mu / g_\tau \approx m_c / m_t \approx s_R / m_R \approx a / b$ one would find that the deviation from maximal mixing is indeed small (cf. eq. (2) with $c = a$) and the neutrinos are almost degenerate in mass, with $m \approx \frac{1}{2} \sqrt{b \Delta m_a^2 / a} \approx 0.35$ eV. We observe that a typical value of $m = b = M_W^2 / M_t$ would imply $M_t \approx 2 \cdot 10^{13}$ GeV whereas $g_\mu g_\tau / m_R \approx \Delta m_s^2 / m$ yields for $g_\tau \approx M_W, g_\mu / g_\tau \approx 10^{-2}$ a value $m_R = 4 \cdot 10^{15}$ GeV. Smaller values of g_τ or m lower the value of m_R .

⁹For a realistic scenario of maximal $\nu_\mu - \nu_\tau$ -mixing it is not sufficient that the leading contributions to the mass matrices have the corresponding structure. The pattern for the smaller (nonleading) masses must remain compatible with maximal mixing.

and nonzero $q = Q$ leaves an $U(1)$ symmetry with $\alpha = -Q/\hat{I}_{3G}$ unbroken. Only the expectation value $\sim M_{\tilde{q}}$ of a second such operator with $\tilde{q} = \tilde{Q}$, $\tilde{Q} \neq 0$ can lead to a mixing between the doublets with different \tilde{q} . The mixing of two doublets whose difference in \tilde{q} is given by $|\Delta\tilde{q}| = p|\tilde{Q}|$ must be proportional $(M_{\tilde{q}})^p$. If $M_{\tilde{q}}$ is smaller than a characteristic common heavy mass M_d (the mass of the heavy doublets), this induces a small parameter $\lambda = M_{\tilde{q}}/M_d$. For $\tilde{Q} = \pm 1$ a typical charged lepton mass pattern for our scenario is

$$M_l = m_\tau \begin{pmatrix} \lambda^4 & \lambda^3 & \lambda^2 \\ \lambda^3 & \lambda^2 & \lambda \\ \lambda^2 & \lambda & 1 \end{pmatrix} \quad (16)$$

where only the order of magnitude is indicated. Similar patterns can arise for up- and down-type quark mass matrices. This is, however, not the topic of this short note. We also repeat here that l^c and ν need not have the same transformation properties with respect to the generation group. (This would only be required for $SO(10)$ unification.) The relation between symmetry and the generation pattern for the charged leptons may be very different from the example discussed above. Nevertheless, the basic observation that the $SU(2)_L$ -doublets and -triplets transform differently under the generation group is rather general. This explains [2] why the generation structure of M_L comes often out quite different from M_l , M_D or M_u .

Despite their different roots in generation symmetry representations and breaking patterns the leading contribution to M_ν and the charged fermion mass matrices show one striking similarity: The two smallest mass eigenvalues are almost degenerate when compared with the scale of the largest mass. This corresponds to an effective invariance of the mass matrix under $U(1)$ -rotations in the plane of the two light eigenstates¹⁰. Whereas for the charged fermions the effective rotations are in the plane of the first two generations, the one in the neutrino sector is in the $\nu_e - \nu_A$ -plane which has an angle of $\pi/4$ with respect to the $\nu_e - \nu_\mu$ -plane. One may call this difference in the planes of effective rotations the “neutrino mismatch”. The question if such a neutrino mismatch can arise naturally is crucial for the hierarchical neutrino mass pattern (A) as well as for the next to leading order contributions in the degenerate pattern (B). For the latter it amounts to the question if a matrix of the type M_{max} (eq. (8)) can be protected by symmetry. We have already identified a candidate for such a symmetry, namely the discrete transformations $R : \nu_e \leftrightarrow \frac{1}{\sqrt{2}}(\nu_\mu + \nu_\tau)$ and $T : \nu_\mu \rightarrow -\nu_\mu, \nu_\tau \rightarrow -\nu_\tau$ (see eq. (11)).

We still have to ask if it is natural that the leading $SU(2)_L$ -triplets (there may be more than one, typically two if $|a| \ll |b|$) are invariant under the transformations R and T . This has to be realized in a natural way in presence

¹⁰These rotations are not necessarily part of a genuine generation symmetry. It may happen that discrete subgroups of the effective rotations belong to the generation symmetry.

of the singlets which induce the doublet mixing responsible for the masses of the first two generations of charged fermions. (These singlets have a fixed value of I_{3G} in our example.) R and T invariance for the induced triplet is obviously realized for $SO(3)_G$ -singlets. It is less trivial for the (perhaps sub-leading) field a responsible for $\Delta m_a^2 \neq 0$. Typically the leading contributions to the $SU(2)_L$ -triplet expectation value $\langle t \rangle$ is induced by a linear term [5] $\sim d_t d_t t(s)$. Here d_t is the expectation value of the leading $SU(2)_L$ -breaking doublet which gives a mass to the top quark. The expectation value of an $SU(2)_L$ -singlet s is needed if $d_t d_t t$ is not invariant under the unification and generation symmetries. In particular, if the unification symmetry includes $B - L$ symmetry, the singlet s has $B - L = -2$. A simple example for a natural neutrino mismatch is a scenario where d_t belongs to a singlet¹¹ of the $SO(3)_G$ -generation symmetry, whereas s breaks $SO(3)_G$, but preserves the discrete subgroup generated by R and T .

The pattern of neutrino mass eigenvalues depends on details of the symmetry-breaking pattern of the generation symmetry. In particular, all three neutrino masses are approximately degenerate if the scale of R - or T -breaking is lower than the breaking of $SO(3)_G$ (or a subgroup enforcing $a = 0$). If this is not realized, a similar size of a and b may lead to a pattern with the mass of two of the neutrinos (ν_e, ν_A) almost degenerate and of the same order but not approximately equal as the mass of the third neutrino ν_B .

Finally, we address the question of the deviation from maximal $\nu_\mu - \nu_\tau$ -mixing or the size of $\epsilon = 1 - \sin^2 2\vartheta$. We assume that the mixing between the second and third generation in the charged lepton mass matrix is small, similar to the quark mass matrices. A value $\vartheta_{23} \approx 0.04$ contributes $\Delta\epsilon_l = 4\vartheta_{23}^2 \approx 6 \cdot 10^{-3}$. In the neutrino sector, a small deviation from M_{max} (8) can be generated from the seesaw mechanism $\sim M_D^T M_R^{-1} M_D$. The terms responsible for a deviation from maximal $\nu_\mu - \nu_\tau$ -mixing are off-diagonal in the standard $SO(3)_G$ -basis where $\tilde{M}_{max} = U^T M_{max} U = \text{diag}(b+a, b+a, b-a)$. They typically also contribute to Δm_s^2 . If the leading correction comes from the element $(\tilde{M}_\nu)_{23}$ (and similar for $(\tilde{M}_\nu)_{13}$), one finds $\Delta m_s^2 = \Delta m_a^2 \Delta\vartheta^2$ or $\Delta\epsilon_\nu = 4\Delta m_s^2 / \Delta m_a^2 \approx 4 \cdot 10^{-3}$. On the other hand, a leading correction in $(\tilde{M}_\nu)_{12}$ would lead to maximal $\nu_e - \nu_A$ -mixing. Although this case would be interesting in its own right, we concentrate here on a small mixing of the electron neutrino and discard this possibility. Combining $\epsilon = (\sqrt{\Delta\epsilon_l} \pm \sqrt{\Delta\epsilon_\nu})^2$ we conclude that the typical deviation from maximal mixing is below 1%!.

It is not our aim here to devise one particular realistic model for a gener-

¹¹It is interesting to note in this context that the first higher dimensional unification model with realistic fermion charges, namely the six-dimensional $SO(12)$ -model [19], exhibits many of the features discussed here if the ground state corresponds to a particular monopole compactification with $SU(5)$ symmetry ($n = 3, m = p = 1$). The generation group $SO(3)_G \times U(1)_G$ has to be broken in the vicinity of the compactification scale. The $SU(2)_L$ -doublet (ν, l) transforms as a triplet under $SO(3)_G$ whereas $(t, b), t^c$ and d_t are singlets.

ation symmetry. We rather want to draw some general conclusions from the outcome of this investigation:

(1) Maximal $\nu_\mu - \nu_\tau$ -mixing can follow naturally from a suitable generation symmetry. It is compatible with a small mass squared difference $|m_{\nu_A}^2 - m_{\nu_e}^2| \approx 5 \cdot 10^{-6} eV^2$. If such a maximal mixing pattern is found experimentally, this would give a strong hint for a nonabelian generation symmetry. A minimal version of such a symmetry can be generated from discrete transformations $R : \nu_e \leftrightarrow \nu_A = \frac{1}{\sqrt{2}}(\nu_\mu + \nu_\tau)$ and $T : \nu_{\mu,\tau} \rightarrow -\nu_{\mu,\tau}$.

(2) Small parameters appear in the deviation of $\sin^2 2\vartheta$ from one. In the neutrino sector the small quantity is given by M_{RT}/M where M_{RT} is a typical scale characterizing the spontaneous breaking of the generation symmetry (i.e. R or T). Typically, this parameter also enters in the mass split between ν_e and ν_A . Discarding a large contribution from the charged lepton mass matrix, the deviation from maximal $\nu_\mu - \nu_\tau$ -mixing is expected to be tiny, $1 - \sin^2 2\vartheta \lesssim 0.01$. This can be viewed as a prediction of natural maximal mixing patterns and clearly distinguishes them from scenarios where the mixing is large but $1 - \sin^2 2\vartheta$ is not related to a small symmetry-breaking scale.

(3) In case of maximal $\nu_\mu - \nu_\tau$ -mixing the leading contribution to the mass matrix for the light neutrinos is presumably due to the expectation value of a heavy $SU(2)_L$ -triplet scalar field rather than to the seesaw-mechanism.

(4) The three neutrino masses can be almost degenerate if the generation group is $SO(3)_G$ with neutrinos belonging to a triplet whereas the leading $SU(2)_L$ -triplet scalar field transforms as a singlet. Maximal $\nu_\mu - \nu_\tau$ -mixing is then explained by the lepton mass matrix not being diagonal in the standard $SO(3)_G$ -basis.

In view of the important implications for an understanding of possible generation symmetries stronger experimental limits for the deviation of $\sin^2 2\vartheta$ from one would be of great value.

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